RANDOM WALKS ON DYADIC-VALUED SOLVABLE MATRIX GROUPS

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Abstract. This paper is concerned with random walks on a family of dyadic-valued solvable matrix groups. A description of the Poisson boundary of these groups for probability measures of finite first moment and *non-zero displacements* (or *drifts*) is given. When non-trivial, the boundary may be identified with a space of matrices with real and 2-adic entries, depending on the values in a *displacement matrix* associated with the random walk. Conditions for boundary triviality are also discussed.

0. Introduction

Baumslag-Solitar groups are the groups $\mathrm{BS}(m,n)$ with the presentations

$$\mathrm{BS}(m,n) = \langle a, t : ta^m t^{-1} = a^n \rangle$$

for natural numbers m and n. The groups $\mathrm{BS}(m,n)$ and $\mathrm{BS}(n,m)$ are isomorphic. They have a linear representation given by the homomorphism induced by

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}.$$

In the case that $|m| = 1$ (or $|n| = 1$), the homomorphism is an isomorphism. In particular, the group $\mathrm{BS}(1,2)$ is isomorphic to the matrix group

$$\mathrm{Aff}\,(\mathbb{Z}[1/2]) = \left\{ \begin{pmatrix} 2^x & f \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}, f \in \mathbb{Z}[1/2] \right\}.$$

where $\mathbb{Z}[1/2]$ denotes the dyadic rationals. Kaimanovich and Vershik [12] gave this group as an example of a solvable group with exponential growth which admits probability measures with non-trivial Poisson boundary.

Let μ be a probability measure of finite first moment on $\mathrm{Aff}(\mathbb{Z}[1/2])$ such that the group generated by $\operatorname{supp}\mu$ is non-abelian and let α be the mean of the image of the measure μ under the homomorphism

$$\begin{pmatrix} 2^k & \frac{m}{2^n} \\ 0 & 1 \end{pmatrix} \mapsto k.$$

Kaimanovich [10] showed that the Poisson boundary—a space associated with every random walk on a locally compact group, see Section 4—is trivial if and only if $\alpha = 0$. He also showed that if α is isomorphic to $\mathbb{R}$ if α is negative, and is isomorphic to $\mathbb{Q}_2$ when α is negative, with the resulting harmonic measure in either case. In the same paper, he stated that "it would be interesting to investigate Poisson boundary of higher-dimensional solvable groups over diadics, for example, groups of triangular matrices" and that "probably the Poisson boundary would consist of both real and

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2-adic components" in that case. Cuno and Sava-Huss [2] have recently provided a description of the Poisson boundary for random walks on Baumslag-Solitar groups $\mathrm{BS}(m,n)$ for $1 < p < q$ for weak conditions on the measure.

In this paper, we discuss the Poisson boundary of the $n \times n$ matrix groups, G_n, whose entries are integer powers of 2 on the main diagonal and dyadic rationals otherwise, generalising the solvable Baumslag-solitar groups. We give the conditions for triviality, and extend Kaimanovich's argument to show that the boundary can indeed contain a mixture of real and 2-adic components when the probability measure associated with the random walk has finite first moment. The techniques developed in this paper are useful for giving descriptions of the asymptotic behaviour of other matrix groups, and semi-direct products of the form $\mathbb{Z}^k \ltimes N$ for nilpotent groups N.

The paper is organised as follows. In Section 1 we discuss random walks on a discrete family of solvable matrix groups, G_n and its decomposition into an external semi-direct product, and give useful formulae for the multiplication of many elements and computation of inverses in the group. We also mention some subgroups and quotients.

In Section 2 we briefly introduce gauge functions and metric estimates on groups. We give generating sets for each matrix group and give a metric estimate which may be efficiently computed for a given group element. We also prove a number of inequalities which will be useful in describing the Poisson boundary.

Section 3 discusses convergence of random walks in each G_n. We define a *mean displacement matrix* (or *drifts*) associated with the random walk, and give conditions for convergence in either the real numbers or the 2-adic numbers. We establish inequalities and bounds on matrix entries in order to do so.

In the Section 4 we briefly give background on the Poisson boundary associated with a random walk. We then discuss the measures for which the boundaries of random walks on each G_n may be trivial.

Finally, in Section 5, we discuss non-trivial boundaries of G_n. A Furstenburg boundary (B, λ) is constructed based on the convergence of entries established in Section. Kaimanovich's Ray Critereon is used to establish the maximality of (B, λ).

1. Preliminaries

For each natural number n, let G_n be the discrete solvable group of upper triangular $n \times n$ matrices whose entries are integer powers of 2 on the main diagonal and dyadic rationals otherwise.

Identify the additive group $\mathbb{Z}^n$ with its isomorphic image under the map

$$(x_1, \cdots, x_n) \mapsto \mathrm{diag}(2^{x_1}, \cdots, 2^{x_n}).$$

in G_n. Write $\boldsymbol{x}$ to mean the n-tuple $(x_1, \cdots, x_n)$ in $\mathbb{Z}^n$.

Let N_n be the subgroup $\mathrm{UT}_n(\mathbb{Z}[1/2])$ of all upper unitriangular matrices contained in G_n. Then, given any matrix f in G_n, there are elements $\boldsymbol{x} = (x_1, \cdots, x_n)$ in $\mathbb{Z}^n$ and $[g]_{ij}$ in N_n where

$$[g]_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \frac{[f]_{ij}}{2^{x_j}} & \text{if } i < j, \\ 0 & \text{if } i > j \end{cases}$$

such that $f = g\boldsymbol{x}$. Hence $G_n = N_n\mathbb{Z}^n$ and we identify G_n with the external semi-direct product $\mathbb{Z}^n \ltimes N_n$. Viewed as this external semi-direct product, the identity element in G_n is (I_n, I_n), where I_n is the $n \times n$ identity matrix. The product of two elements $(\boldsymbol{x}, f)$ and $(\boldsymbol{y}, g)$ in G_n is

$$(\boldsymbol{x}, f)(\boldsymbol{y}, g) = (\boldsymbol{x}+\boldsymbol{y}, f\boldsymbol{x}g\boldsymbol{x}^{-1}).$$

The product of m elements,

$$(\boldsymbol{y}^{(m)}, \varphi^{(m)}) \coloneqq \prod_{i=1}^{m}(\boldsymbol{x}^{(i)}, f^{(i)}) \coloneqq (\boldsymbol{x}^{(1)}, f^{(1)}) \cdot \ldots \cdot (\boldsymbol{x}^{(m)}, f^{(m)}),$$

is given by the relations

$$\begin{aligned} y_i^{(k)} &= \sum_{r=1}^{k} x_i^{(r)}; \\ \varphi^{(m)} &= \varphi^{(m-1)}\boldsymbol{y}^{(m-1)} f^{(m)}(-\boldsymbol{y}^{(m-1)}), \end{aligned} \tag{1}$$

where $\varphi^{(1)} = f^{(1)}$. Since every entry on the diagonal of $f^{(m)}$ is 1, and $[\varphi^{(l)}]_{pq} = 0$ for all $p \le q$ and $l \in \mathbb{Z}$,

$$\begin{aligned} [\varphi^{(m)}]_{ij} &= \sum_{k=1}^{n}[\varphi^{(m-1)}]_{ik}[f^{(m)}]_{kj}2^{y_k^{(m-1)}-y_j^{(m-1)}} \\ &= \sum_{\substack{k=1\\k\neq j}}^{n}[\varphi^{(m-1)}]_{ik}[f^{(m)}]_{kj}2^{y_k^{(m-1)}-y_j^{(m-1)}} + [\varphi^{(m-1)}]_{ij} \\ &= \sum_{l=1}^{m-1}\left(\sum_{k=i}^{j-1}[\varphi^{(l)}]_{ik}[f^{(l+1)}]_{kj}2^{y_k^{(l)}-y_j^{(l)}}\right) + [f^{(1)}]_{ij} \\ &= \sum_{k=i}^{j-1}\sum_{l=0}^{m-1}[\varphi^{(l)}]_{ik}[f^{(l+1)}]_{kj}2^{y_k^{(l)}-y_j^{(l)}} \end{aligned} \tag{2}$$

whenever $j > i$ and $\varphi^{(0)}$ is taken to be the identity. Expanded entirely,

$$[\varphi^{(m)}]_{i,i+r} = \sum_{\{a_k\}\in P(r)} S^{(m)}_{\{a_k\}} \tag{3}$$

where

$$S^{(m)}_{\{a_k\}} = \sum_{0\le b_1<\ldots<b_{|\{a_k\}|-1}<m} \prod_{n=0}^{|\{a_k\}|-2} [f^{(b_{n+1}+1)}]_{i+a_n,i+a_{n+1}} 2^{y_{i+a_n}^{(b_{n+1})}-y_{i+a_{n+1}}^{(b_{n+1})}}.$$

The validity of this formula may be checked by induction on equation (2).

The next proposition gives a useful explicit formula for the inverse of any upper unitriangular matrix, which is useful for calculating inverses of elements $(\boldsymbol{x}, f)$ in G_n, because

$$(\boldsymbol{x}, f)^{-1} = (\boldsymbol{x}^{-1}, \boldsymbol{x}^{-1}f^{-1}\boldsymbol{x}).$$

Proposition 1.0.1 (Unitriangular matrix inverse)**.** *Suppose that f is an $n \times n$ upper unitriangular matrix. Then, for each pair of natural numbers i and s so that*

$i < n$ and $s \leq n - i$, the equation

$$[f^{-1}]_{i,i+s} = \sum_{l=1}^{s} (-1)^l \left(\sum_{\{h_\zeta\} \in H(l,s)} \left(\prod_{\xi=0}^{l-1} [f]_{i+h_\xi, i+h_{\xi+1}} \right) \right)$$

is satisfied, where $H(l,s)$ is the collection of finite sequences of integers $\{h_0, \cdots, h_l\}$ so that $h_k < h_{k+1}$, $h_1 = 0$ and $h_l = s$.

Proof. Consider the system of linear equations $ff^{-1} = I_n$. This system states that

$$0 = \sum_{k=1}^{n} [f]_{ik} [f^{-1}]_{k,i+s}$$

whenever $l > i$. If i is a natural number less than n, then

$$[f]_{ii} = 1$$

and if k is a natural number so that $k > s$, then

$$[f^{-1}]_{i+k,i+s} = 0.$$

Consequently,

$$[f^{-1}]_{i,i+s} = -\sum_{k=1}^{s} [f]_{i,i+k} [f^{-1}]_{i+k,i+s} \tag{4}$$

whenever i and s are natural numbers so that $i < n$ and $s < n-1$. We now proceed by inducting on s. The proposition is true for $s = 1$, because

$$[f^{-1}]_{i,i+1} = -[f]_{i,i+1}$$

for all natural numbers i so that $i < n$. Suppose that equation (4) is satisfied for some integer s so that $s \geq r$. Then, as

$$\begin{aligned}
&[f^{-1}]_{i,i+s+1} \\
&\quad = -\sum_{k=1}^{s+1} [f]_{i,i+k} [f^{-1}]_{i+k,i+s+1} \\
&\quad = -\sum_{k=1}^{s} [f]_{i,i+k} [f^{-1}]_{i+k,i+s+1} - [f]_{i,i+s+1} \\
&\quad = -\sum_{k=1}^{s} [f]_{i,i+k} \sum_{l=1}^{s-k+1} (-1)^l \left(\sum_{\{h_\zeta\} \in H(l,s-k+1)} \left(\prod_{\xi=0}^{l-1} [f]_{i+k+h_\xi, i+k+h_{\xi+1}} \right) \right) \\
&\qquad - [f]_{i,i+s+1} \\
&\quad = \sum_{l=1}^{s+1} (-1)^l \left(\sum_{\{h_\zeta\} \in H(l,s+1)} \left(\prod_{\xi=0}^{l-1} [f]_{i+h_\xi, i+h_{\xi+1}} \right) \right),
\end{aligned}$$

equation (4) is satisfied for $s + 1$. This means that our formula for f^{-1} gives us at least a right inverse for f. If g is the left inverse of f then it is also a left inverse of f, as

$$g = gI_n = gff^{-1} = I_n f^{-1} = f^{-1}$$

which completes the proof. $\square$

1.1. **The difference subgroup of G_n.** Suppose that K_n is any subgroup of N_n and let F_n be the subgroup $\mathbb{Z}^n \ltimes K_n$ of G_n. Let $\Delta\colon \mathbb{Z}^n \to H_{n-1}$ be the group homomorphism

$$\Delta(x_1, \cdots, x_n) = (x_1 - x_2, \cdots, x_{n-1} - x_n).$$

We call Δ the *difference homomorphism*. In appropriate contexts we will also use this term to denote the map from F_n to $H_{n-1} \times K_n$ given by

$$\Delta(\boldsymbol{x}, f) = (\Delta(\boldsymbol{x}), f).$$

In either case, Δ is surjective, and it splits; in particular if $s\colon H_{n-1} \to \mathbb{Z}^n$ is the map

$$s(d_1, \cdots, d_{n-1}) = \left(0, -d_1, -d_1 - d_2, \cdots, -\sum_{i=1}^{n-1} d_i\right)$$

then $\Delta \circ s$ is the identity mapping on F_n and if Δ is the map from $H_{n-1} \times K_n$ to F_n given by

$$\Delta(\boldsymbol{d}, f) = (s(\boldsymbol{d}), f)$$

then $\Delta \circ s$ is the identity mapping on $\mathbb{Z}^n$. Let D_n be the semi-direct product of H_{n-1} acting on K_n under the action induced by s:

$$(d_1, \cdots, d_n) \cdot f = s(d_1, \cdots, d_n) \cdot f.$$

We shall call $D(F_n)$ the *difference subgroup of* F_n. The kernel of $\Delta\colon \mathbb{Z}^n \to H_{n-1}$ is

$$\ker \Delta = \{(x_1, \cdots, x_n) \in \mathbb{Z}^n \colon x_i = x_{i+1}, f_{i,i+1} = 0 \ \forall i \in \{1, \cdots, n-1\}\}$$

and $F_n / \ker \Delta$ is isomorphic to the image of Δ, $D(F_n)$.

As Δ splits, the following is a short exact sequence:

$$\{e\} \to \ker \Delta \to F_n \rightleftarrows F_n / \ker \Delta \to \{e\}$$

which implies that F_n is isomorphic to a semi-direct product of $\ker \Delta$ acting on $F_n / \ker \Delta$. This product is direct, because $\ker \Delta$ has trivial action on $F_n / \ker \Delta$. If the random walk is given by a product measure on $(\ker \Delta) \times F_n / \ker \Delta$ then recurrence arguments may be applied to show that the Poisson boundaries of $F_n / \ker \Delta$ and F_n can be identified.

It is also possible to describe the Poisson boundary of $F_n / \ker \Delta$ with respect to any probability measure satisfying certain weak conditions directly. We do so in section 5.2. The argument is similar to the one given for describing the boundary of G_n under similar restrictions on the probability measure.

1.2. **Subgroups and quotients of G_n.** The group G_n is solvable because it is a semi-direct product of an abelian group acting on a nilpotent one. In particular, G_n has a subnormal series

$$G_n = G_n^{(0)} \triangleright G_n^{(1)} \triangleright \cdots \triangleright G_n^{(k)} \triangleright \cdots \triangleright G_n^{(n)} = \{e\},$$

where

$$
\begin{aligned}
G_n^{(0)} &= G_n, \\
G_n^{(1)} &= [G_n, G_n] = \{I_n\} \times N_n \cong N_n = N_n^{(0)}, \\
G_n^{(2)} &= [G_n^{(1)}, G_n^{(1)}] = [\{I_n\} \times N_n, \{I_n\} \times N_n] \cong [N_n, N_n] = N_n^{(1)}, \\
&\vdots\ \vdots \\
G_n^{(k)} &= [G_n^{(k-1)}, G_n^{(k-1)}] = [\{I_n\} \times N_n^{(k-2)}, \{I_n\} \times N_n^{(k-2)}] \cong N_n^{(k-1)}, \\
&\vdots \\
G_n^{(n)} &= \{e\}.
\end{aligned}
$$

Suppose that k is an integer. Then $N_n^{(k)}$ is the set of $n \times n$ upper unitriangular matrices for which the entires in the first k upper super-diagonals are all zero. In particular, $N_n^{(n-1)}$ contains only the identity matrix I_n. Obviously, $G_n^{(k)}$ is nilpotent whenever $k \geq 1$. We also note that $(G_n/\ker\Delta)^{(k)}$ and $G_n^{(k)}$ are isomorphic as groups for all natural numbers k (excluding $k = 0$).

The first quotient group $G_n/G_n^{(1)}$ is isomorphic to the additive group $\mathbb{Z}^n$ which is abelian. The quotients $G_n/G_n^{(k)}$ for $k \geq 1$ are the cosets of G_n for which any two elements (x, f) and (y, g) are identified if $x = y$ and any differences in f and g occur in entries above the first k upper super-diagonals. These groups are solvable, but not nilpotent, for $k > 1$. If j is a natural number so that $j < k$, then the quotient $G_n^{(k)}/G_n^{(j)}$ is isomorphic to $N_n^{(k)}/N_n^{(j)}$, which is nilpotent.

In subsection 5.2, we will briefly discuss the Poisson boundary of the solvable quotients $G_n/G_n^{(k)}$ for every natural number k so that $k > 1$. The remaining subgroups and quotient groups mentioned here are nilpotent and hence uninteresting from that point of view because the Poisson boundary is trivial for any probability measure—see the discussion in section 4 .

2. Metric estimates

Let G be a locally compact group with a compact symmetric generating set K. A *gauge function*—in the sense used by Kaimanovich [10, 11]—is a non-negative real-valued function δ for which there exists a positive constant C such that

$$\delta(g_1 g_2) \leq \delta(g_1) + \delta(g_2) + C$$

for all g_1 and g_2 in G. The word length function

$$|g|_K = \min\{n \in \mathbb{N} : g \in K^n\},$$

with respect to a generating set K on a group G, is an example of a gauge function on G. Suppose that δ_1 and δ_2 are non-negative real-valued functions on G satisfying

$$A_1\delta_1(g) - B_1 \leq \delta_2(g) \leq A_2\delta_1(g) + B_2 \tag{5}$$

for some real constants $A_1, A_2 > 0$ and $B_1, B_2 \geq 0$. Then δ_1 is a gauge function if and only if δ_2 is a gauge function. If δ_1 and δ_2 are gauge functions satisfying equation (5) then they are *equivalent*.

If K' is some other compact symmetric generating set, then $|\cdot|_K$ and $|\cdot|_{K'}$ are equivalent gauge functions—consider the maximum K' word length of each

generator in K and vice versa. A gauge function is a *metric estimate* or *principle gauge function* if is equivalent to a word length gauge function on G.

We now exhibit a generating set for G_n, and define a gauge function $[\![\cdot]\!]$ on G_n which may be computed directly from the matrix entries of the element. We will then show that this gauge function is a metric estimate.

Let J_n be the subset of G_n consisting of the elements $e_p = (\delta_{pp} + I_n, I_n)$ and $e_{pq} = (I_n, \delta_{pq} + I_n)$ for natural numbers p and q so that $p < q \leq n$, where

$$[\delta_{pq}]_{ij} = \begin{cases} 1 & \text{if } p = i, q = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $K_n = J_n \cup J_n^{-1}$ and let $|\cdot|_{K_n} : G_n \to \mathbb{Z}^{\geq 0}$ be the corresponding word length.

Lemma 2.0.1. *The finite set K_n is a symmetric generating set for G_n.*

Proof. Suppose that $(\boldsymbol{x}, f)$ is an element in G_n so that $[f]_{ij} = \frac{m_{ij}}{2^{n_{ij}}}$ where m_{ij} and n_{ij} are integers for all natural numbers i and j such that $i < j$ and so that $2 \nmid m_{ij}$. It is well known that the set

$$\{\delta_{pq} + I_n, \delta_{pq}^{-1} + I_n : p, q \in \mathbb{N}, p < q \leq n\}$$

is a generating set for the group of integer valued upper triangular matrices with ones on the main diagonal—see e.g. Elder, Elston and Ostheimer [3]. Consequently, there is an integer valued matrix $z \in \mathrm{UT}(\mathbb{Z}) \leq N_n$ so that

$$(\boldsymbol{x}, f) = \left(\prod_{i=1}^{n} e_i^{-x_i c}\right) \left(\prod_{j>i}^{n} e_{ij}^{[z]_{ij}}\right) \left(\prod_{i=1}^{n} e_i^{x_i}\right),$$

where c is the value of the largest $[n]_{ij}$. $\square$

We are now ready to construct a metric estimate on each G_n. As per Kaimanovich [10], for each dyadic rational f, let

$$d_+(f) = \max\{i : \epsilon_i = 1\},$$

$$d_-(f) = \min\{i : \epsilon_i = 1\}$$

and let

$$\|f\| = \begin{cases} 1 + \max\{|d_-(f)|, |d_+(f)|\} & \text{if } f \neq 0, \\ 0 & \text{if } f = 0 \end{cases}$$

where $\sum_{i=k}^{r} \epsilon_i 2^i$ is the unique binary representation of f.

If f and g are dyadic rationals, then

$$\begin{aligned} |d_+(fg)| &\leq \log_2 |fg| \\ &= \log_2 |f| + \log_2 |g| \\ &\leq 2 + |d_+(f)| + |d_+(g)| \end{aligned}$$

and it follows from consideration of binary decimal expansions that

$$d_-(fg) = d_-(f) + d_-(g).$$

Lemma 2.0.2 (Additive triangle inequality)**.** *If $f, g \in \mathbb{Z}[1/2]$ then*

$$\|f + g\| \leq \|f\| + \|g\|.$$

Proof. Recall from the definition of $\|\cdot\|$ that

$$\|f+g\| = \begin{cases} 1+\max\{|d_-(f+g)|, |d_+(f+g)|\} & \text{if } f+g \neq 0, \\ 0 & \text{if } f+g = 0. \end{cases}$$

If $f+g$ is zero, then the result is clear. Suppose that $f+g \neq 0$. If

$$|d_+(f+g)| \geq |d_-(f+g)|,$$

then by considering addition in terms of the binary expansions of f and g,

$$\begin{aligned} \|f+g\| &\leq 1+|d_+(f+g)| \\ &\leq 1+|\max(|d_+(f)|, |d_+(g)|)+1 \\ &\leq 1+|d_+(f)|+1+|d_+(g)| \\ &\leq \|f\|+\|g\| \end{aligned}$$

and similarly if

$$|d_+(f+g)| < |d_-(f+g)|,$$

then

$$\begin{aligned} \|f+g\| &\leq 1+|d_-(f+g)| \\ cv &\leq 1+\max(|d_-(f)|, |d_-(g)|) \\ &\leq 1+|d_-(f)|+1+|d_-(g)| \\ &\leq \|f\|+\|g\| \end{aligned}$$

which completes the proof. □

Lemma 2.0.3 (Multiplication bound)**.** *If $f, g \in \mathbb{Z}[1/2]$ then*

$$\|fg\| \leq 3(\|f\|+\|g\|).$$

Proof. The statement is true if $f=0$ or $g=0$. Suppose then that $f \neq 0$ and $g \neq 0$. Then

$$\begin{aligned} \|fg\| &= 1+\max\{|d_-(fg)|, |d_+(fg)|\} \\ &\leq 1+\max\{2+|d_-(f)|+|d_-(g)|, 2+|d_+(f)|+|d_+(g)|\} \\ &\leq 3+\max\{|d_-(f)|+|d_-(g)|, |d_+(f)|+|d_+(g)|\} \\ &\leq 3+2\max\{|d_-(f)|, |d_+(f)|\}+2\max\{|d_-(g)|, |d_+(g)|\} \\ &\leq 3(\|f\|+\|g\|) \end{aligned}$$

which completes the proof.

□

Proposition 2.0.4 (Metric estimate on G_n)**.** *For each element $(\boldsymbol{x}, f)$ in G_n, let*

$$[\![(\boldsymbol{x}, f)]\!] = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \|[f]_{ij}\|,$$

where $|z|$ is the ordinary absolute value of an integer z. Then there are positive real constants B_n and C_n (dependent on n) so that

$$B_n[\![(\boldsymbol{x}, f)]\!] \leq |(\boldsymbol{x}, f)|_{K_n} \leq C_n[\![(\boldsymbol{x}, f)]\!],$$

i.e., the gauge function $[\![\cdot]\!]$ is a metric estimate on G.

Proof. If f is the identity, then

$$|(\boldsymbol{x}, f)|_{K_n} = \sum_{i=1}^{n} |x_i| = [\![(\boldsymbol{x}, f)]\!].$$

Suppose that f is not the identity. The statement of the lemma is true for $n = 1$ as

$$|g|_{K_1} = [\![g]\!].$$

Let k be a natural number and suppose that for every $g_k \in G_k$ that

$$|g_k|_{K_{k+1}} \leq C_k [\![g_k]\!].$$

Let $(\boldsymbol{x}, f)$ be in G_{k+1} and assume, without loss of generality, that $(\boldsymbol{x}, f)$ is not the identity. Let h and g be the $n \times n$ matrices

$$[h]_{ij} = \begin{cases} [f]_{ij} & \text{if } \ i = 1, \\ 1 & \text{if } \ i = j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$[g]_{ij} = \begin{cases} [f]_{ij} & \text{if } \ i \neq 1, \\ 1 & \text{if } \ i = j, \\ 0 & \text{otherwise} \end{cases}.$$

Identify G_k with the subgroup in G_{k+1} consisting of all elements $(\boldsymbol{z}, h) \in G_{k+1}$ for which $z_1 = 0$ and $[h]_{ij} = 0$ whenever $i = 1$ and $j > 1$ and observe that

$$\begin{aligned} |(\boldsymbol{x}, f)|_{K_{k+1}} &= |(\boldsymbol{x}, gh)|_{K_{k+1}} \\ &= |(\boldsymbol{x}, gI_n h I_n^{-1})|_{K_{k+1}} \\ &\leq |(I_n, g)|_{K_{k+1}} + |(\boldsymbol{x}, h)|_{K_{k+1}} \\ &\leq C_k [\![(I_n, g)]\!] + |(\boldsymbol{x}, h)|. \end{aligned}$$

Let $2^{n_j} \sum_{k=0}^{t_j} \epsilon_k^j 2^k$ be the unique binary decimal expansion of $[h]_{ij}$ for each natural number j not equal to 1 and less than or equal to n. Then

$$(\boldsymbol{x}, h) = \left(\prod_{j=1}^{n} e_j^{x_j} \right) \left(\prod_{j=2}^{n} e_1^{-n_j} \left(\prod_{i=0}^{t_j} \gamma_k^j e_1 \right) e_1^{n_j - t_j - x_1} \right), \tag{6}$$

where

$$\gamma_k^j = \begin{cases} e & \text{if } \epsilon_k^j = 0, \\ e_{1j} & \text{if } \epsilon_k^j = 1. \end{cases}$$

Since $(\boldsymbol{x}, f)$ is not the identity, equation (6) gives that

$$\begin{aligned}
|(\boldsymbol{x},h)|_{K_{k+1}} \leq \sum_{j=2}^{n} |d_-([h]_{1j})| &+ 2(d_+([h]_{1j} - d_-([h]_{1j}) + 1) \\
&\qquad + |x_1 - d_+([h]_{1j})| + \sum_{j=1}^{n} |x_j| \\
&\leq (n-1) + \sum_{j=1}^{n} |x_j| + \sum_{j=2}^{n} |x_1| + 3\sum_{j=2}^{n} \left(|d_+([h]_{1j})| + |d_-([h]_{1j})|\right) \\
&\leq 3(n-1)\left(\sum_{j=1}^{n} |x_j| + \sum_{j=1}^{n} \|[h]_{1j}\|\right),
\end{aligned}$$

so that

$$|(\boldsymbol{x}, f)|_{K_{k+1}} \leq 3(n-1)C_k[\![(\boldsymbol{x}, f)]\!].$$

The image of each generator e_p under the homomorphism $(\boldsymbol{x}, f) \mapsto x_p$ is 1, which implies that

$$\sum_{i=1}^{n} |x_i| \leq n|(\boldsymbol{x}, f)|_{K_n}.$$

It then follows from the recurrence relation in equation (1) for $\varphi^{(m)}$, that

$$|d_-([f]_{ij})| \leq |(\boldsymbol{x}, f)|_{K_n}$$

and that

$$|[f]_{ij}| \leq 2^{|(x,f)|_{K_n}}$$

hence

$$|d_+([f]_{ij})| \leq \log_2 |[f]_{ij}| \leq |(\boldsymbol{x}, f)|_{K_n}.$$

Since f is not the identity by assumption, $1 \leq |(\boldsymbol{x}, f)|_{K_n}$. Thus, as there are $\frac{(n-1)(n-2)}{2}$ upper semi-direct entries in f,

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \|[f]_{ij}\| \leq \left(\frac{n^2 - 3n + 2}{2}\right) |(\boldsymbol{x}, f)|_{K_n}$$

which means that

$$\left(\frac{2}{n^2 - 2n + 2}\right) [\![(\boldsymbol{x}, f)]\!] \leq |(\boldsymbol{x}, f)|_{K_n}$$

which completes the proof. □

3. Convergence of random walks

For appropriate probability measures, random walks on G_n converge almost surely to matrices with entries in the $\mathbb{Q}_2$ (the 2-adics), or $\mathbb{R}$, depending on the sign of a *mean displacement matrix* associated with each walk. In this section, we will prove this. We begin with a brief introduction to random walks on groups, so that we may fix our terminology and notation.

A random walk is a pair (G, μ) where G is a locally compact group and μ is a probability measure on G over the Borel σ-algebra, $\mathscr{B}(G)$. Each random walk is

identified with the discrete time-homogeneous Markov chain which has state space G and transition probabilities

$$p(g, A) = \mu(g^{-1}A).$$

The transition probabilities are *group invariant* in the sense that for all group elements g in G and measurable sets A_1 and A_2 in $\mathscr{B}(G)$ the probability of transition from A_1 to A_2 is the same as the probability of transition from gA_1 to gA_2. The initial distribution is taken to be the point mass at the identity unless otherwise specified.

The *support* of the measure μ, denoted $\operatorname{supp}(\mu)$ is the complement of the union of all open sets E such that $\mu(E) = 0$. We denote the closed semi-group generated by the support of μ by

$$\operatorname{sgr}\mu = \overline{\bigcup_{n\geq 0} \operatorname{supp}\mu^n}.$$

The measure μ and the random walk (G, μ) are both said to be *non-degenerate* if $\operatorname{sgr}\mu$ is G. The measure μ is *finitary* if the support of μ is finite. The *reflection* of μ is the measure $\check{\mu}$ given by

$$\check{\mu}(E) = \mu(E^{-1}).$$

for each measurable set E. The measure μ and the random walk (G, μ) are both said to be *symmetric* if $\mu = \check{\mu}$. The measure μ is *discrete* if the support of μ is a countable set.

If $f\colon X_1 \to X_2$ is a measurable mapping between a measure space (X_1, μ_1) and a measurable space X_2 then the *pushforward* or *image* of the measure μ_1 with respect to f is the measure

$$f_*\mu_1 = \mu_1 \circ f^{-1}.$$

If μ_1 is a probability measure, then $f_*\mu_1$ is a probability measure.

The *path space* or *space of trajectories* is the countably infinite Cartesian product

$$G^\infty = \prod_{i\in\mathbb{N}} G.$$

A *trajectory* or *path* is any sequence of group elements $y = (y_0, y_1, \ldots)$ in G^∞.

Let $\mathbb{P}^\mu$ be the image of the product measure on G^∞ given by the map

$$(x_1, x_2, x_3, \cdots, x_k, \cdots) \mapsto (x_1, x_1x_2, x_1x_2x_3, \cdots, x_1 \cdot \ldots \cdot x_k, \cdots)$$

with the product sigma algebra. The pair $(G^\infty, \mathbb{P}^\mu)$ is called the *path space*. For brevity, we often write $\mathbb{P}^\mu$–*a.s.* to mean $\mathbb{P}^\mu$–almost surely and $\mathbb{P}^\mu$–a.s. to mean $\mathbb{P}^\mu$–almost everywhere.

If G is a compactly generated discrete group, then a measure μ on G has *finite first moment* if

$$\int \delta(g)\, d\mu(g) < +\infty$$

for some metric estimate δ on G — see section 2.

3.1. **Random walks on G_n.** From now on, we suppose that μ is a measure on G_n with a finite first moment such that the group generated by $\operatorname{supp}\mu$ is non-abelian and that

$$(\boldsymbol{y}^{(m)}, \varphi^{(m)}) = \prod_{i=1}^{m} (\boldsymbol{x}^{(m)}, f^{(m)})$$

is a path in the random walk associated with μ on G_n.

Definition 3.1.1. For every natural number p less than n, let Π_p be the map from $\mathbb{Z}^n \ltimes N_n$ to $\mathbb{Z}$ given by

$$\Pi_p(\boldsymbol{x}, f) = x_p,$$

let μ_p be the push forward measure given by

$$\mu_p(E) = {\Pi_p}_* \mu$$

and let $\overline{\mu_p} \coloneqq \sum_{z \in \mathbb{Z}} z \mu_p$ be the corresponding mean of each measure μ_p.

Similarly, for natural numbers p and q less than n, let Π_{pq} be the map from G_n to $\mathbb{Z}[1/2]$ given by

$$\Pi_{pq}(\boldsymbol{x}, f) = [f]_{pq}$$

and let μ_{pq} be the corresponding push forward measure:

$$\mu_{pq}(E) = {\Pi_{pq}}_* \mu.$$

Finally, let $\overline{\mu_{pq}} \coloneqq \int_{\mathbb{Z}[1/2]} z \, d\mu_{pq}$ be the mean of μ_{pq}.

Lemma 3.1.2. *The mean of each measure $\overline{\mu_p}$ is finite.*

Proof. Because the first moment of μ is finite, Theorem 12.1 in Schilling [15] implies that

$$\begin{aligned} \overline{\mu_p} &= \int_{\mathbb{Z}} z \, d{\Pi_p}_* \mu(z) \\ &\leq \int_{\mathbb{Z}} |z| \, d{\Pi_p}_* \mu(z) \\ &= \int_{G_n} |\Pi_p(z)| \, d\mu(z) \\ &\leq \int_{G_n} |\Pi_p(z)|_{K_n} \, d\mu(z) \end{aligned}$$

which is finite since $|\cdot|_{K_n}$ is equivalent to every principle gauge function on G_n. □

Definition 3.1.3. Let D be the matrix in $\mathrm{UT}_n(\mathbb{R})$ given by

$$[D]_{ij} = \begin{cases} \overline{\mu_i} - \overline{\mu_j} & \text{if } i \leq j \\ 0 & \text{if } i \geq j. \end{cases}$$

We will refer to D as the *mean displacement matrix associated with* (G_n, μ). Each super-diagonal entry $[D]_{ij}$ shall be referred to as a *mean displacement associated with* (G_n, μ). We shall say that (G_n, μ) is of *non-zero mean displacement*, *all negative displacement* or *all positive displacement* if each displacement is non-zero, strictly negative or strictly positive respectively.

If i, j and k are natural numbers so that $i \leq j \leq k \leq n$, then

$$[D]_{ik} = [D]_{ij} + [D]_{jk}.$$

Furthermore, if $\{a_k\}$ is an element of $P(r)$—as defined for equation (3), then

$$[D]_{i,i+r} = \sum_{k=0}^{|a_k|-1} [D]_{a_k, a_{k+1}}.$$

In particular—taking $\{a_k\} = \{0, 1, \cdots, j-i\}$ in this expression—D is determined exactly by its entries on the first super-diagonal.

The following lemma is similar to one given by Kaimanovich [10]. We provide a proof for completeness.

Lemma 3.1.4. *For every pair of natural numbers i and j which are both less than or equal to n,*

$$\log\left(1+|[f^{(m)}]_{ij}|\right)\in o(m) \quad \mathbb{P}^{\mu}\text{–}a.s.$$

and

$$\log\left(1+|[f^{(m)}]_{ij}|_2\right)\in o(m) \quad \mathbb{P}^{\mu}\text{–}a.s.$$

Proof. Let i and j be natural numbers so that $i<j\le n$. Then

$$\begin{aligned}
|\log(1+|[f^{(m)}]_{ij}|)| &\le |d_+(1+|[f^{(m)}]_{ij}|)| \\
&\le 1+|d_+([f^{(m)}]_{ij})| \\
&\le 1+\max\{|d_-([f^{(m)}]_{ij})|,|d_+([f^{(m)}]_{ij})|\} \\
&\le \|[f^{(m)}]_{ij}\| \\
&\le \frac{1}{B_n}|(\boldsymbol{x}^{(m)},f^{(m)})|_{K_n}
\end{aligned}$$

for some positive real constant B_n. Similarly,

$$\begin{aligned}
|\log(1+|[f^{(m)}]_{ij}|_2)| &\le 1+|d_-([f^{(m)}]_{ij})| \\
&\le 1+\max\{|d_-([f^{(m)}]_{ij})|,|d_+([f^{(m)}]_{ij})|\} \\
&\le \|[f^{(m)}]_{ij}\| \\
&\le \frac{1}{B_n}|(\boldsymbol{x}^{(m)},f^{(m)})|_{K_n}.
\end{aligned}$$

Because the first moment of μ is finite,

$$|(\boldsymbol{x}^{(m)},f^{(m)})|_{K_n}\in o(m) \quad \mathbb{P}^{\mu}\text{–a.s.}$$

because if we suppose to the contrary then the integral

$$\int_{G_n}|(\boldsymbol{x},f)|_{K_n}\,d\mu(\boldsymbol{x},f)$$

is almost surely not bounded. This completes the proof. □

Lemma 3.1.5. *Suppose that i and j are natural numbers less than or equal to n. Then, there are positive real constants $C(i,j)$ and a natural number N so that*

$$|[\varphi^{(m)}]_{ij}|\le C(i,j)m^{2(j-i)}2^{m[D]_{ij}} \ \ \mathbb{P}^{\mu}\text{–}a.s. \tag{7}$$

whenever $m>N$.

Proof. We will proceed by induction. It is clear that $1=[\varphi^{(m)}]_{ii}$. Let p be a natural number and suppose that equation (7) is satisfied whenever $j<p$. Let

$$[g^{(l)}]_{ij}=\sum_{r=0}^{l}[f^{(r)}]_{ij}$$

for each pair of natural numbers i and j less than or equal to n. Then,

$$\begin{aligned}
|[\varphi^{(m)}]_{ip}| &\le \sum_{k=i}^{p-1}\sum_{l=0}^{m-1} |[\varphi^{(l)}]_{ik}||[f^{(l+1)}]_{kp}|2^{y_k^{(l)}-y_p^{(l)}} \\
&\le \sum_{k=i}^{p-1}\sum_{l=0}^{m-1} l^{2(k-i)}C(i,k)2^{l[D]_{ik}}|[f^{(l+1)}]_{kj}|2^{l[D]_{kp}}2^{y_k^{(l)}-y_p^{(l)}-l[D]_{kp}} \\
&= \sum_{k=i}^{p-1}\sum_{l=0}^{m-1} l^{2(k-i)}C(i,k)|[f^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_p^{(l)}-l[D]_{kp}}2^{l([D]_{ik}+[D]_{kp})} \\
&= \sum_{k=i}^{p-1}\sum_{l=0}^{m-1} l^{2(k-i)}C(i,k)|[f^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_p^{(l)}-l[D]_{kp}}2^{l[D]_{ip}} \\
&\le \sum_{k=i}^{p-1}\sum_{l=0}^{m-1} l^{2(k-i)}C(i,k)|[g^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_p^{(l)}-l[D]_{kp}}2^{l[D]_{ip}} \\
&\le \sum_{k=i}^{p-1}\sum_{l=0}^{m-1} l^{2(p-1-i)}C(i,k)|[g^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_p^{(l)}-l[D]_{kp}}2^{l[D]_{ip}} \\
&\le \sum_{k=i}^{p-1} m^{2(p-1-i)}\sum_{l=0}^{m-1} C(i,k)|[g^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_p^{(l)}-l[D]_{kp}}2^{l[D]_{ip}} \ \mathbb{P}^{\mu}\text{–a.s.}
\end{aligned}$$

The strong law of large numbers implies that

$$\lim_{l\to\infty}\frac{1}{l}\left(y_k^{(l)}-y_p^{(l)}\right)=[D]_{kp} \text{ and } \lim_{l\to\infty}\frac{1}{l}[g^{(l+1)}]_{kj}=|\mu_{kj}| \ \mathbb{P}^{\mu}\text{–a.s.,}$$

which implies that

$$|[g^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_p^{(l)}-l[D]_{kp}} \in o(l) \ \mathbb{P}^{\mu}\text{–a.s.}$$

Hence, there is a positive constant $C(i,p)$ and a natural number N satisfying

$$\begin{aligned}
|[\varphi^{(m)}]_{ip}| &\le C(i,p)m^{2(p-1-i)}\sum_{l=0}^{m-1} l^2\, 2^{l[D]_{ip}} \\
&\le C(i,p)m^{2(p-i)}\sum_{l=0}^{m-1} 2^{l[D]_{ip}} \\
&= C(i,p)m^{2(p-i)}\left(2^{m[D]_{ip}}-1\right) \\
&\le C(i,p)m^{2(p-i)}2^{m[D]_{ip}}.
\end{aligned}$$

for all $m > M$. □

Remark 3.1.6. If $[D]_{ij} < 0$, then $2^{m[D]_{ij}}$ converges almost surely to 1. Consequently, the bound in the statement of the lemma may be simplified to

$$|[\varphi^{(m)}]_{ij}| \le C(i,j)m^{2(j-i)}.$$

In the next proposition, we show that the negative displacement condition, $[D]_{ij} < 0$, implies convergence—and thus boundedness—of $[\varphi^{(m)}]_{ij}$.

Proposition 3.1.7. *Suppose that i and j are natural numbers less than or equal to n. Each sequence $[\varphi^{(m)}]_{ij}$ converges almost surely in $\mathbb{R}$ provided that the corresponding displacement, $[D]_{ij}$, is less than 0.*

Proof. Recall equation (2) which stated that

$$[\varphi^{(m)}]_{ij} = \sum_{k=i}^{j-1}\sum_{l=0}^{m-1}[\varphi^{(l)}]_{ik}[f^{(l+1)}]_{kj}2^{y_k^{(l)}-y_j^{(l)}}.$$

If $j = i + 1$, then this equation becomes

$$[\varphi^{(m)}]_{ij} = \sum_{l=0}^{m-1}[f^{(l+1)}]_{ij}2^{y_i^{(l)}-y_j^{(l)}}.$$

By the strong law of large numbers,

$$\lim_{l\to\infty}\frac{1}{l}\left(y_i^{(l)} - y_j^{(l)}\right) = [D]_{ij}.$$

Since the sequence $|[f^{(l+1)}]_{ij}|$ almost surely grows subexponentially—see Lemma 3.1.4—the series converges almost surely by the ratio test.

From now on, suppose that $j > i + 1$. As absolute convergence implies convergence, it is sufficient to show that for any integer k satisfying $i \le k < j$ each summation

$$\sum_{l=0}^{m-1}|[\varphi^{(l)}]_{ik}||[f^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_j^{(l)}} \tag{8}$$

is almost surely convergent in $\mathbb{R}$. Using the bound on the growth of each sequence $|[\varphi^{(l)}]_{ik}|$ from Lemma 3.1.5, where k is an integer so that $i < k \le n$, and the strong law of large numbers,

$$\begin{aligned}\sum_{l=0}^{m-1}|[\varphi^{(l)}]_{ik}||[f^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_j^{(l)}} &\le \sum_{l=0}^{m-1}C(i,k)l^{2(k-i)}2^{l[D]_{ik}}|[f^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_j^{(l)}}\\ &= \sum_{l=0}^{m-1}C(i,k)l^{2(k-i)}|[f^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_k^{(l)}-l[D]_{kj}}2^{l([D]_{ik}+[D]_{kj})}\\ &\le \sum_{l=0}^{m-1}C(i,k)l^{2(k-i)}|[g^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_k^{(l)}-l[D]_{kj}}2^{l[D]_{ij}}.\end{aligned}$$

As per the argument in Lemma 3.1.5,

$$|[g^{(l+1)}]_{kj}|2^{y_k^{(l)}-y_k^{(l)}-l[D]_{kj}} \in o(l).$$

It follows from our supposition that the displacement $[D]_{ij}$ is negative. Since, for each natural number k so that $i < k \le n$, the sequence $|[f^{(l+1)}]_{kj}|$ almost surely grows subexponentially (see Lemma 3.1.4), the series in equation (8) is $\mathbb{P}^\mu$–a.s. convergent because it is monotone increasing and bounded above by a $\mathbb{P}^\mu$–a.s. convergent series. ☐

Corollary 3.1.8. *Suppose that (G_n, μ) is of all negative displacements. Then the sequence of matrices $\varphi^{(m)}$ will converge pointwise $\mathbb{P}^\mu$–a.s. to a matrix $\varphi^{(\infty)}$ in $\mathrm{UT}_n(\mathbb{R})$.*

Proposition 3.1.9. *Suppose that i and j are natural numbers which are both less than or equal to n. Each sequence $[\varphi^{(m)}]_{ij}$ converges $\mathbb{P}^{\mu}$–a.s. in $\mathbb{Q}_2$ provided that the displacement $[D]_{ij}$ is greater than 0.*

Proof. We proceed by considering the formula given in equation (3) for sufficiently large m. We must show that for each $\{a_k\} \in P(j-i)$ that

$$S^{(m)}_{\{a_k\}} = \sum_{0 \le b_1 < \ldots < b_{|\{a_k\}|-1} < m} \prod_{n=0}^{|\{a_k\}|-2} [f^{(b_{n+1}+1)}]_{i+a_n, i+a_{n+1}} 2^{y^{(b_{n+1})}_{i+a_n} - y^{(b_{n+1})}_{i+a_{n+1}}}$$

converges $\mathbb{P}^{\mu}$–a.s. in $\mathbb{Q}_2$ as $m \to \infty$. Suppose that $\{a_k\}$ is the length two sequence $\{0, j-i\}$. In this case,

$$S^{(m)}_{\{a_k\}} = \sum_{l=0}^{m-1} [f^{(l+1)}]_{i,i+1} 2^{y^{(l)}_i - y^{(l)}_{i+1}}.$$

By Lemma 3.1.4, the sequence $\left|[f^{(l+1)}]_{i,i+1}\right|_2$, grows subexponentially, hence, the strong law of large numbers implies that

$$\lim_{l\to\infty} \left|[f^{(l+1)}]_{ij} 2^{y^{(l)}_i - y^{(l)}_j}\right|_2 = \lim_{l\to\infty} \left|[f^{(l+1)}]_{ij}\right|_2 2^{-l[D]_{ij}} = 0.$$

The 2-adic norm is non-archimedian, so this is sufficient to show that the sequence of partial sums $S^{(m)}_{\{a_k\}}$ is $\mathbb{P}^{\mu}$–a.s. Cauchy and hence $\mathbb{P}^{\mu}$–a.s. convergent with respect to that norm.

Suppose that $\{a_k\}$ is some other element of $P(j-i)$. Then $\{a_k\}$ has length of at least 3. A series $\sum_{i=1}^{n} a_i$ converges in $\mathbb{Q}_p$ if the sequence $\{a_n\}$ converges in $\mathbb{Q}_p$. Using this fact recursively (a finite number of times), the sequence $S^{(m)}_{\{a_k\}}$ converges whenever the sequence

$$U^{(m)}_{\{a_k\}} = \prod_{n=0}^{|\{a_k\}|-2} [f^{(b_{n+1}+1)}]_{i+a_n, i+a_{n+1}} 2^{y^{(b_{n+1})}_{i+a_n} - y^{(b_{n+1})}_{i+a_{n+1}}},$$

where

$$\begin{aligned} b_{|\{a_k\}|-1} &= m, \\ b_{|\{a_k\}|-2} &= m-1, \\ &\vdots \\ b_1 &= m - (|\{a_k\}| - 2). \end{aligned}$$

is convergent. To show that $U^{(m)}_{\{a_k\}}$ is convergent $\mathbb{P}^{\mu}$–a.s. in $\mathbb{Q}_2$, note that

$$\begin{aligned} U^{(m)}_{\{a_k\}} &= \prod_{n=0}^{|\{a_k\}|-2} [f^{(b_{n+1}+1)}]_{i+a_n, i+a_{n+1}} 2^{y^{(b_{n+1})}_{i+a_n} - y^{(b_{n+1})}_{i+a_{n+1}} - b_{n+1}[D]_{i+a_n, i+a_{n+1}}} 2^{b_{n+1}[D]_{i+a_n, i+a_{n+1}}} \\ &= 2^{m[D]_{i,j}} \prod_{n=0}^{|\{a_k\}|-2} [f^{(b_{n+1}+1)}]_{i+a_n, i+a_{n+1}} 2^{y^{(b_{n+1})}_{i+a_n} - y^{(b_{n+1})}_{i+a_{n+1}} - b_{n+1}[D]_{i+a_n, i+a_{n+1}}} \prod_{l=-|\{a_k\}|+2}^{0} 2^{l[D]_{i,j}}. \end{aligned}$$

Following the argument in Lemma 3.1.7, which gave conditions for $[\varphi^{(m)}]_{ij}$ to converge $\mathbb{P}^{\mu}$–a.s. in $\mathbb{R}$,

$$2^{y^{(l)}_{a_n} - y^{(l)}_{a_{n+1}} - l[D]_{a_n, a_{n+1}}} \in o(l)$$

for each integer n satisfying $0 \leq n < |\{a_k\}| - 1$. This means that the 2-adic norm of the first product,

$$\left| \prod_{n=0}^{|\{a_k\}|-2} [f^{(b_{n+1}+1)}]_{i+a_n, i+a_{n+1}} 2^{y^{(b_{n+1})}_{i+a_n} - y^{(b_{n+1})}_{i+a_{n+1}} - b_{n+1}[D]_{i+a_n, i+a_{n+1}}} \right|_2,$$

grows subexponentially with b_{n+1} a.s.. The second product is a finite rational constant. As $[D]_{i,j}$ is positive, we conclude that

$$\lim_{m\to\infty} \left| U^{(m)}_{\{a_k\}} \right|_2 = 0 \quad \mathbb{P}^\mu\text{–a.s.},$$

hence that $U^{(m)}_{\{a_k\}}$ converges $\mathbb{P}^\mu$–almost surely in $\mathbb{Q}_2$. □

Corollary 3.1.10. *Suppose that (G_n, μ) is of all positive displacements. Then the sequence of matrices $\varphi^{(m)}$ will converge pointwise $\mathbb{P}^\mu$–a.s. to a matrix $\varphi^{(\infty)}$ in $\mathrm{UT}_n(\mathbb{Q}_2)$.*

4. Conditions for Poisson boundary triviality

There are many equivalent definitions of the Poisson boundary of a random walk. In this section, we will characterise the measures of finite first moment on G_n which have trivial Poisson boundary. We will begin by giving a brief account of the theory. See Erschler [6], Furstenberg [7, 8], or Kaimanovich and Vershik [12] for a more detailed account.

Suppose that G is a second countable locally compact group and let μ be a probability measure on G. A μ-measurable complex-valued function f is *μ-harmonic* if it satisfies

$$f(x) = f * \mu(x) = \int_G f(xy)\, d\mu(y)$$

for almost every $x \in G$. The C^*-algebra of all essentially bounded μ-harmonic functions with multiplication

$$(f_1 \cdot f_2)(x) = \lim_{n\to\infty} \int f_1(xy) f_2(xy)\, d\mu^n(y)$$

and the complex conjugation involution $f \mapsto \bar{f}$ in $L^\infty(G)$ is denoted $H^\infty_\mu(G)$. See e.g. Erschler [5], Kaimanovich and Vershik [12] or Willis [16] for more details.

If G is a topological group and X is a topological space, then the pair $(X, \cdot)$ is a *G-space* if G acts on X continuously. If, in addition, for each fixed $g \in G$ the map $X \to X$, $x \mapsto g \cdot x$ is measurable, then $(X, \cdot)$ is a *measurable G-space*.

The *Poisson space*, Π_μ, corresponding to μ is the spectrum of the commutative C^*-algebra $H^\infty_\mu(G)$. The spectrum is endowed with a probability measure ν which is *μ-stationary*, i.e. it satisfies the convolution equation $\mu * \nu = \nu$. The measure ν is called the *Poisson kernel* and satisfies

$$\int \hat{f}(x)\, d\nu(x) = f(e)$$

where $\hat{f}$ is the Gelfand transform of f. The maps $f(x) \mapsto f(gx)$, for each g in G, form an automorphism group on $C(\Pi)$. By a theorem of Nagata [13], there is a corresponding group of homeomorphisms on Π, allowing a group action to be defined on Π so that it is a measurable G-space. The *Poisson boundary* of (G, μ) is

the pair (Π, ν). Every essentially bounded μ-harmonic function f admits a *Poisson representation*

$$f(g) = \int \hat{f}(x)\, dg\nu(x) = \int f(x) \frac{dg\nu}{d\nu}(x)\, d\nu(x),$$

where $g\nu(E) := \nu(g^{-1}E)$ and $\hat{f}$ is an essentially bounded ν-measurable function on Π.

Suppose that $\pi : G^\infty \to B$ is a map from the path space G^∞ to a measurable G-space B, and suppose that λ is a measure on B which is an image of the measure $\mathbb{P}^\mu$ on G^∞. In addition, suppose that π is measurable with respect to the stationary σ-algebra on G^∞ and that π is G-equivariant, that is $\pi(gx) = g\pi(x)$, where $(gy)_n = gy_n$. Then (B, λ) is a *Furstenberg boundary* of (G, μ). The Poisson boundary is maximal in the sense that every Furstenberg boundary of (G, μ) is an equivariant image of the Poisson boundary.

For some groups, the Poisson boundary is trivial for every non-degenerate choice of probability measure. This is the case for e.g. nilpotent groups, because the only bounded harmonic functions are constant—see e.g. Chu and Hilberdink [1]. If μ is a symmetric probability measure with a finite first moment on a polycyclic group G—a group G is *polycyclic* if it admits a series of normal subgroups $G = G_n \triangleright \cdots \triangleright G_2 \triangleright G_1 \triangleright G_0 = \{0\}$ so that all of the factor groups G_{i+1}/G_i are cyclic—then the Poisson boundary is also trivial [10]. It has been shown by Rosenblatt [14], and later Kaimanovich and Vershik [12] that if $\operatorname{supp}\mu$ generates a non-amenable group, then the Poisson boundary is non-trivial.

Kaimanovich and Vershik [12] gave examples of solvable groups with non-degenerate probability measures such that the boundary is non-trivial. Rosenblatt [14] and later Kaimanovich and Vershik [12] showed that every countable, amenable group has a non-degenerate symmetric probability measure such that the boundary of the corresponding random walk is trivial.

Erschler [4] showed that if G is a finitely generated solvable group then it admits a symmetric measure with non-trivial Poisson boundary if and only if the group is not virtually nilpotent. This may be combined with Gromov's theorem [9] to show that G is a finitely generated solvable group then it admits a symmetric measure with non-trivial Poisson boundary if and only if the group does not have polynomial growth. Finding a symmetric probability measure on a group G for which the Poisson boundary is non-trivial can therefore be used to give a lower bound on the growth of G.

4.1. **Recurrence.** A subset R of G is a *recurrent set* if it satisfies either of the following equivalent conditions (see e.g. Woess [17] for a proof):

(1) The probability of a walk eventually returning to R is 1, and
(2) The expected number of visits to R is infinite.

A subset of G which does not satisfy either of these conditions is called a *transient set*. We mention a number of results about recurrent subgroups and boundaries of semi-direct products. See Furstenberg [8], Kaimanovich and Vershik [12] or Woess [17] for more information.

If (G, μ) is a random walk so that a subgroup G_0 is a recurrent set, then it is possible to identify the Poisson boundaries of (G, μ) and (G_0, μ_0) if we define a *hitting measure* μ_0 on G_0 appropriately. The following definitions and lemmas are given with this goal in mind.

Given a measure ν on G and a measurable subset E, let $\nu|_E(B) = \nu(B \cap E)$ be the measure which is the *restriction of* ν *on* E.

Let (G, μ) be a group with measure and let G_0 be a measurable subgroup which is a recurrent set. Define the sequence of measures

$$\mu^{(0)} = \mu;$$
$$\mu^{(k+1)} = \mu|_R^{(k)} + \left(\mu^{(k)} - \mu|_R^{(k)}\right) * \mu$$

Each $\mu^{(k)}$ is a probability measure and as R is a recurrent set,

$$\lim_{k\to\infty} \mu^{(k)}(R) = 1$$

and

$$\lim_{k\to\infty} \mu^{(k)}(G \setminus R) = 0.$$

It follows that the sequence $\mu^{(k)}$ is Cauchy in total variation distance. The space $\mathscr{P}(G)$ of all probability measures on G is complete because it is a closed subspace of the Banach space $M(G)$. It follows that $\mu^{(k)}$ has a limit, a probability measure μ_0 called the *hitting measure* on G_0.

Proposition 4.1.1. *Let G be a discrete group, μ a probability measure on G, and G^0 be a subgroup of G which is a recurrent set for the random walk (G, μ). Let μ^0 be the hitting measure on G^0. Then the Poisson boundaries associated with (G, μ) and (G_0, μ_0) are isomorphic.*

Proof. See Furstenberg [8] or Kaimanovich [10]. □

4.2. **Boundary triviality in G_n.** In this section we give necessary and sufficient conditions for probability measures μ on G_n to have a trivial Poisson boundary.

Lemma 4.2.1. *Suppose that μ is a probability measure on G_n. Let Π_{pq} be the map from G_n to $\mathbb{Z}[1/2]$ where*

$$\Pi_{pq}(x, f) = [f]_{pq}$$

and let

$$(\boldsymbol{y}^{(m)}, \varphi^{(m)}) = \prod_{i=1}^{m} (\boldsymbol{x}^{(m)}, f^{(m)})$$

be a path in the random walk (G_n, μ). Suppose that

$$\operatorname{sgr}\left(\Pi_{pq*}\mu(G_n)\right) = \{0\}$$

for natural numbers p and q so that $p < q \leq n$. Then

$$[\varphi^{(m)}]_{ij} = [f^{(m)}]_{ij} = 0$$

$\mathbb{P}^\mu$–a.s. for all integers i and j satisfying $p \leq i < j \leq q$.

Proof. Suppose that $[f^{(m)}]_{ij} \neq 0$ for some natural numbers i and j satisfying $p \leq i < j \leq q$ and some natural number m. It follows from equation (3) that our supposition is in contradiction with the assumption in the statement of the lemma. □

Proposition 4.2.2. *Suppose that μ is a probability measure on G_n. The Poisson boundary of (G, μ) is trivial if and only if for every pair of natural numbers p and q satisfying $p < q \leq n$ at least one of the following conditions is satisfied:*

(1) $[D(\mu)]_{pq} = 0$, *or*
(2) $\operatorname{sgr}\left(\Pi_{pq_*}\mu(G_n)\right) = \{0\}$.

In particular, if μ is of zero mean displacement or $\operatorname{sgr}\mu$ *generates an Abelian group then the Poisson boundary of (G_n, μ) is trivial.*

Proof. Suppose that p and q are a pair of natural numbers so that $p < q \le n$, and at least one of the two conditions is satisfied. Let R_n be the subgroup of $\mathbb{Z}^n$ given by

$$R_n = \{(x_1, \cdots, x_n) : x_i = x_j \ \forall i, j \text{ such that } [D]_{ij} = 0\}.$$

Then $G_0 = R_n \ltimes Nn$ is a subgroup which is recurrent in $\mathbb{Z}^n \ltimes N_n$. In light of Lemma 4.2.1, our multiplication formula in equation (3) and because of the condition imposed on the support μ, the action of R_n is trivial on N_n in this subgroup. Consequently G_0 is isomorphic to a nilpotent group (the direct product of two nilpotent groups is nilpotent. Let μ_0 be the hitting measure on G_0. As G_0 is nilpotent, the Poisson boundary of (G_n, μ_0) is trivial. As G_0 is a recurrent subgroup, Corollary 4.1.1 states that the boundaries (G_0, μ_0) and (G_n, μ_0) are isomorphic, where μ_0 is the hitting measure on G_0, hence that the boundary of (G_n, μ) is also trivial.

Suppose that there are natural numbers p and q so that $[D(\mu)]_{pq} \neq 0$ and $\operatorname{sgr}\left(\Pi_{pq_*}\mu(G_n)\right) \neq \{0\}$. Then, the group generated by $\operatorname{supp}\mu$ is non-abelian, and there must exist a pair of points in $\operatorname{supp}\mu$ which have commutator (I_n, h) where $[h]_{pq}$ is non-zero.

Suppose for contradiction that the Poisson boundary is trivial, i.e. that the limit $\varphi^{(\infty)}$ is the same for almost every pair of paths $(\boldsymbol{y}^{(m)}, \varphi^{(m)})$. Since $[D(\mu)]_{pq} \neq 0$, it is the case that $[\varphi^{(\infty)}]_{pq} \neq 0 \ \ \mathbb{P}^{\mu}$–a.s.. Considering the action of the commutator (I_n, h) on the supposedly unique boundary point $\varphi^{(\infty)}$, we see that

$$(I_n, h) \cdot \varphi^{(\infty)} = h I_n \varphi^{(\infty)} I_n^{-1} = \varphi^{(\infty)}.$$

which is a contradiction, because it implies that $[h]_{pq} = 0$. □

If all the displacements in the random walk are zero. Then it is also possible to show that the boundary is trivial with an induction argument. We give this argument in the next lemma as a matter of interest.

Proposition 4.2.3. *Let μ be a probability measure on G_n. Suppose that each random walk $(\mathbb{Z}, \mu_{x_i})$ is recurrent for all natural numbers i so that $i \le n$. Then the Poisson boundary of (G_n, μ) is trivial*

Proof. We begin by identifying G_k with the subgroup in G_n consisting of elements $(\boldsymbol{x}, f)$ so that $x_r = 0$ whenever $r > k$ and $[f]_{ij} = 0$ whenever $j > k$ and $j \neq i$. Then each subgroup G_{k-1} is a recurrent set in G_k for the random walk associated with (G_k, μ_k), where $\mu_n = \mu$ and μ_{k-1} is the hitting measure of (G_k, μ_k) on G_{k-1}. For this reason we may identify the Poisson boundary of (G_n, μ) with the Poisson boundary of (G_1, μ_1) by recursively applying lemma 4.1.1 to the subgroups G_k. But G_1 is abelian because it is isomorphic to $\mathbb{Z}$, so it has trivial boundary. □

5. Non-trivial boundaries

Suppose that μ is a measure on G_n with a finite first moment such that the boundary of (G_n, μ) is non-trivial—see Proposition 4.2.2 for the conditions required.

Let

$$(\boldsymbol{y}^{(m)}, \varphi^{(m)}) = \prod_{i=1}^{m} (\boldsymbol{x}^{(m)}, f^{(m)})$$

be a path in the random walk associated with μ. Let D be the associated mean displacement matrix, as in definition 3.1.3. Let B be the set of all $n \times n$ matrices M whose entries $[M]_{ij}$ are respectively in $\mathbb{R}$ or $\mathbb{Q}_2$ if $[D]_{ij} < 0$ or $[D]_{ij} > 0$. Given a path

$$(\boldsymbol{y}^{(m)}, \varphi^{(m)}) = \prod_{i=1}^{m} (\boldsymbol{x}^{(m)}, f^{(m)}),$$

in the random walk, let $\varphi^{(\infty)}$ be the matrix in B with entries

$$[\varphi^{(\infty)}]_{ij} = \lim_{m\to\infty} [\varphi^{(m)}]_{ij}.$$

The set B is a G-space under the action of left multiplication, in particular if

$$\lim_{m\to\infty} \varphi^{(m)} = \varphi^{(\infty)}$$

then

$$\lim_{m\to\infty} (\boldsymbol{x}, f) \cdot \varphi^{(m)} = (\boldsymbol{x}, f) \cdot \varphi^{(\infty)}.$$

Let λ be the hitting measure of (G, μ) on B. Then (B, λ) is a Furstenberg boundary of (G, μ). To show that (B, λ) is maximal, we use the following well known theorem of Kaimanovich [10] (or [11] for a more general statement), combined with the metric estimate we obtained in Section 2.

Theorem 5.0.4 (Kaimanovich's Ray Criterion)**.** *Let μ be a probability measure on a finitely generated group G, and (B, λ) be a Furstenberg boundary. Let d be the word length metric corresponding to some finite generating set on G. If there exists a sequence of measurable maps $\pi_m \colon B \to G$ such that*

$$\frac{1}{m} d(\pi_m(z_\infty), z_m) \to 0$$

for $\mathbb{P}^\mu$–a.e. path $z = z_m$ in G^∞, where z_∞ is the point in B corresponding to the path z and d is the wordlength metric, then (B, λ) is the Poisson boundary of the pair (G, μ).

Roughly speaking, to show that B is maximal, we must define a sequence of measurable maps π_m from B to G_n so that for every path $(\boldsymbol{y}^{(m)}, \varphi^{(\infty)})$ so that

$$\lim_{m\to\infty} \varphi^{(m)} = \varphi^{(\infty)} \in B,$$

the sequence $\pi_m \varphi^{(\infty)}$ approximates $(\boldsymbol{y}^{(m)}, \varphi^{(\infty)})$ 'well enough' in the sense of Kaimanovich's ray criterion.

Given real numbers x and p, let the *right truncation* T_R of x with respect to p be

$$T_R\,(x, p) = 2^{-p} \lfloor 2^p x \rfloor$$

where $\lfloor \cdot \rfloor$ is the floor function. Let the *left truncation* T_L of x with respect to p be

$$T_L\,(x, p) = 2^p \operatorname{frac}(2^{-p} x).$$

where $\operatorname{frac}(z) = z - \lfloor z \rfloor$ is the fractional part of z. We will write $T_R x$ to mean $T_R(x, 0) = \lfloor x \rfloor$ and $T_L x$ to mean $T_L(x, 0) = \lfloor x \rfloor$. For each $m \in \mathbb{N}$ let $T^{(m)} \colon B \to \mathrm{UT}_n(\mathbb{Z}[1/2])$ be the map

$$[T^{(m)}\varphi^{(\infty)}]_{ij} = \begin{cases} T_R\left([\varphi^{(\infty)}]_{ij}, m\right) & \text{if } [\varphi^{(m)}]_{ij} \text{ converges in } \mathbb{R}, \\ T_L\left([\varphi^{(\infty)}]_{ij}, m\right) & \text{if } [\varphi^{(m)}]_{ij} \text{ converges in } \mathbb{Q}_2. \end{cases}$$

We now define our maps $\pi_m \colon B \to G_n$ for each $m \in \mathbb{N}$. Given $\varphi^{(\infty)}$, let

$$\pi_m \varphi^{(\infty)} = (\boldsymbol{t}^{(m)}, T^{(m)}\varphi^{(\infty)}),$$

where $\boldsymbol{t}^{(m)} = (T_R m\overline{\mu_{x_1}}, \cdots, T_R m\overline{\mu_{x_n}})$. Then

$$(\pi_m\varphi^{(\infty)})^{-1} = (\boldsymbol{t}^{(m)^{-1}}, \boldsymbol{t}^{(m)^{-1}}(T^{(m)}\varphi^{(\infty)})^{-1}\boldsymbol{t}^{(m)})$$

which means that

$$\begin{aligned}\left(\pi_m\varphi^{(\infty)}\right)^{-1}(\boldsymbol{y}^{(m)}, \varphi^{(m)}) &= ((\boldsymbol{t}^{(m)})^{-1}\boldsymbol{x}^{(m)}, (\boldsymbol{t}^{(m)})^{-1}(T^{(m)}\varphi^{(\infty)})^{-1}\boldsymbol{t}^{(m)}(\boldsymbol{t}^{(m)})^{-1}\varphi^{(m)}\boldsymbol{t}^{(m)}) \\ &= ((\boldsymbol{t}^{(m)})^{-1}\boldsymbol{x}^{(m)}, (\boldsymbol{t}^{(m)})^{-1}(T^{(m)}\varphi^{(\infty)})^{-1}\varphi^{(m)}\boldsymbol{t}^{(m)}) \\ &= (\boldsymbol{t^{-1}}^{(m)}\boldsymbol{x}^{(m)}, \gamma^{(m)}),\end{aligned}$$

where $\gamma^{(m)}$ is defined to be $(\boldsymbol{t}^{(m)})^{-1}(T^{(m)}\varphi^{(\infty)})^{-1}\varphi^{(m)}\boldsymbol{t}^{(m)}$ for brevity. From the bound given in lemma 2.0.4, there is a positive real constant C_n—not dependant on the particular path—so that

$$\begin{aligned}|\left(\pi_m\varphi^{(\infty)}\right)^{-1}(\boldsymbol{y}^{(m)}, \varphi^{(m)})|_{K_n} &= \left|((\boldsymbol{t}^{(m)})^{-1}\boldsymbol{x}^{(m)}, \gamma^{(m)})\right|_{K_n} \\ &\leq C_n [\![((\boldsymbol{t}^{(m)})^{-1}\boldsymbol{x}^{(m)}, \gamma^{(m)})]\!] \\ &\leq C_n\left(\sum_{i=1}^{n}|x_i^{(m)} - T_R m\overline{\mu_{x_i}}| + \sum_{i=1}^{n-1}\sum_{r=1}^{n-i}\|[\gamma^{(m)}]_{i,i+r}\|\right).\end{aligned}$$

Lemma 5.0.5. *For each natural number i less than n,*

$$|x_i^{(m)} - T_R(m\overline{\mu_i})| \in o(m) \quad \mathbb{P}^{\mu}\text{–}a.s.$$

Proof. By lemma 3.1.2 the mean of each measure $\overline{\mu_i}$ is finite. The result then follows from strong law of large numbers. □

In light of this lemma, (B, λ) would be maximal provided that

$$\|[\gamma^{(m)}]_{i,i+r}\| \in o(m)$$

for all natural numbers i and r so that $i < n$ and $r < n - i$. We will show that this is the case, by inducting on r. We consider the base case, $r = 1$, in the next lemma. The argument is similar to one made by Kaimanovich in [10] whilst describing the Poisson boundary of $\mathrm{Aff}(\mathbb{Z}[1/2])$.

Lemma 5.0.6. *Let γ as above, and let i be a natural number less than n. Then,*

$$\|[\gamma^{(m)}]_{i,i+1}\| \in o(m) \quad \mathbb{P}^{\mu}\text{–}a.s.$$

Proof. Using the formula given in Proposition 1.0.1,

$$[T^{(m)}\varphi^{(\infty)}]^{-1}_{i,i+1} = -[T^{(m)}\varphi^{(\infty)}]_{i,i+1}.$$

As we have assumed boundary non-triviality for this section, each $[D]_{i,i+1}$ is finite and non-zero. Using the definition of matrix multiplication,

$$[\gamma^{(m)}]_{i,i+1} = \; 2^{-T_R(l[D]_{i,i+1})}\left([\varphi^{(m)}]_{i,i+1} - [T^{(m)}\varphi^{(m)}]_{i,i+1}\right),$$

where, using equation (2),

$$[\varphi^{(m)}]_{i,i+1} = \sum_{l=0}^{m-1}[f^{(l+1)}]_{i,i+1}2^{y_i^{(l)}-y_{i+1}^{(l)}}.$$

We first consider the case where $[D]_{i,i+1} < 0$. Then,

$$\begin{aligned}
[\gamma^{(m)}]_{i,i+1} =& 2^{-T_R(l[D]_{i,i+1})}\left(\sum_{l=0}^{m-1}[f^{(l+1)}]_{i,i+1}2^{y_i^{(l)}-y_{i+1}^{(l)}}\right) \\
& - 2^{-T_R(l[D]_{i,i+1})}\left(T_R\left(\sum_{l=0}^{\infty}[f^{(l+1)}]_{i,i+1}2^{y_i^{(l)}-y_{i+1}^{(l)}}, -T_R(l[D]_{i,i+1})\right)\right). \\
=& \sum_{l=0}^{m-1}[f^{(l+1)}]_{i,i+1}2^{y_i^{(l)}-y_{i+1}^{(l)}-T_R(l[D]_{i,i+1})} - \left\lfloor \sum_{l=0}^{\infty}[f^{(l+1)}]_{i,i+1}2^{y_i^{(l)}-y_{i+1}^{(l)}-T_R(l[D]_{i,i+1})}\right\rfloor.
\end{aligned}$$

The strong law of large numbers implies that

$$\lim_{m\to\infty}\frac{1}{m}\left(y_i^{(m)} - y_{i+1}^{(m)}\right) = [D]_{i,i+1},$$

and since the sequence $|[f^{(l+1)}]_{i,i+1}|$ $\mathbb{P}^\mu$–a.s. grows subexponentially (lemma 3.1.4), $d_-\left([f^{(m+1)}]_{i,i+1}\right) \in o(m)$ $\mathbb{P}^\mu$–a.s. We may therefore conclude that

$$\begin{aligned}
d_-([\gamma^{(m)}]_{i,i+1}) &\geq \min\left(0, d_-\left(\sum_{l=0}^{m-1}[f^{(l+1)}]_{i,i+1}2^{y_i^{(l)}-y_{i+1}^{(l)}-T_R(l[D]_{i,i+1})}\right)\right) \\
&\geq \min_{0\leq l<m}\left(0, d_-\left([f^{(l+1)}]_{i,i+1}2^{y_i^{(l)}-y_{i+1}^{(l)}-T_R(l[D]_{i,i+1})}\right)\right) \\
&= \min_{0\leq l<m}\left(0, y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1}) + d_-\left([f^{(l+1)}]_{i,i+1}\right)\right) \in o(m) \;\mathbb{P}^\mu\text{–a.s.}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|[\gamma^{(m)}]_{i,i+1}| = & \left| \sum_{l=0}^{m-1} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} - \left\lfloor \sum_{l=0}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \right\rfloor \right| \\
\leq & 1 + \left| \sum_{l=0}^{m-1} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \right. \\
& \left. - \left\lfloor \sum_{l=0}^{m-1} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \right\rfloor - \left\lfloor \sum_{l=m}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \right\rfloor \right| \\
\leq & 2 + \left| \left\lfloor \sum_{l=m}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \right\rfloor \right| \\
\leq & 3 + \left| \sum_{l=m}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \right| \\
\leq & 3 + \sum_{l=m}^{\infty} \left| [f^{(l+1)}]_{i,i+1} \right| 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})}.
\end{aligned}$$

Once again using strong law of large numbers and that $|[f^{(l+1)}]_{i,i+1}|$ almost surely grows subexponentially, for each natural number i, it follows that

$$d_+([\gamma^{(m)}]_{i,i+1}|) \in o(m) \ \mathbb{P}^\mu\text{–a.s.}$$

Hence, using the definition of $\|\cdot\|$, we may conclude that

$$\|[\gamma^{(m)}]_{i,i+1}\| \in o(m) \ \mathbb{P}^\mu\text{–a.s.}$$

whenever $[D]_{i,i+1} < 0$.

Suppose now instead that $[D]_{i,i+1} > 0$. Then

$$\begin{aligned}
[\gamma^{(m)}]_{i,i+1} = & 2^{-T_R(l[D]_{i,i+1})} \left(\sum_{l=0}^{m-1} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)}} \right) \\
& - 2^{-T_R(l[D]_{i,i+1})} \left(T_L \left(\sum_{l=0}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)}}, -T_R(l[D]_{i,i+1}) \right) \right). \\
= & \sum_{l=0}^{m-1} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} - \operatorname{frac} \left(\sum_{l=0}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \right).
\end{aligned}$$

Since the absolute value of the fractional part of any number is at most 1, similar arguments apply for the negative displacement case by using the strong law of large numbers and the bound on the growth of the sequence $|[f^{(l+1)}]_{i,i+1}|$ given in Proposition 3.1.4. We begin by letting $[g^{(l)}]_{ij} = \sum_{r=0}^{l} [f^{(r)}]_{ij}$ for every pair of integers i and j less than n. Then we observe that

$$\lim_{l\to\infty} \frac{1}{l} [g^{(l+1)}]_{kj} = |\mu_{kj}|.$$

Then we can conclude that

$$\begin{aligned}
|[\gamma^{(m)}]_{i,i+1}| \leq & 1 + \sum_{l=0}^{m-1} |[f^{(l+1)}]_{i,i+1}| 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \\
\leq & 1 + \sum_{l=0}^{m-1} |[g^{(l+1)}]_{i,i+1}| 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})} \in o(m).
\end{aligned}$$

Finally, using similar arguments to as before,

$$\begin{aligned}
d_-\left([\gamma^{(m)}]_{i,i+1}\right) =& d_-\left(\sum_{l=0}^{m-1} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})}\right. \\
& \left. - \operatorname{frac}\left(\sum_{l=0}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})}\right)\right) \\
\geq & \min\left(0, d_-\left(\sum_{l=m}^{\infty} [f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})}\right)\right) \\
\geq & \min_{m \leq l}\left(0, d_-\left([f^{(l+1)}]_{i,i+1} 2^{y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1})}\right)\right) \\
\geq & \min_{m \leq l}\left(0, y_i^{(l)} - y_{i+1}^{(l)} - T_R(l[D]_{i,i+1}) + d_-\left([f^{(l+1)}]_{i,i+1}\right)\right) \in o(m).
\end{aligned}$$

It is now evident that $\|[\gamma^{(m)}]_{i,i+1}\| \in o(m)$ whenever $[D]_{i,i+1} \neq 0$. $\square$

We can compute $[\gamma^{(m)}]_{i,i+r}$ using the definition of matrix multiplication:

$$[\gamma^{(m)}]_{i,i+r} = 2^{-T_R(m[D]_{i,i+r})} \sum_{s=0}^{r} [T^{(m)}\varphi^{(m)}]^{-1}_{i,i+s} [\varphi^{(m)}]_{i+s,i+r}.$$

As $[T^{(m)}\varphi^{(\infty)}]^{-1}_{i,i} = 1$, we have that

$$\begin{aligned}
[\gamma^{(m)}]_{i,i+r} =& 2^{-T_R(m[D]_{i,i+r})} [\varphi^{(m)}]_{i,i+r} \\
& + 2^{-T_R(m[D]_{i,i+r})} \sum_{s=1}^{r} [T^{(m)}\varphi^{(m)}]^{-1}_{i,i+s} [\varphi^{(m)}]_{i+s,i+r}.
\end{aligned}$$

From Proposition 1.0.1, we have the following formula for every pair of natural numbers i and s so that $i < n$ and $s < n - i$:

$$[T^{(m)}\varphi^{(\infty)}]^{-1}_{i,i+r} = \sum_{j=1}^{r} \left((-1)^j \sum_{\{h_\zeta\} \in H(j,r)} \left(\prod_{\xi=0}^{j-1} [T^{(m)}\varphi^{\infty}]_{i+h_\xi, i+h_{\xi+1}} \right) \right).$$

In particular, this formula tells us that when $s \geq 1$, we have the following recurrence relation

$$\begin{aligned}
[T^{(m)}\varphi^{(\infty)}]^{-1}_{i,i+s} &= -\sum_{k=1}^{s} \left([T^{(m)}\varphi^{(\infty)}]_{i,i+k} [T^{(m)}\varphi^{(\infty)}]^{-1}_{i+k,i+s} \right); \\
[T^{(m)}\varphi^{(\infty)}]^{-1}_{i,i+1} &= -[T^{(m)}\varphi^{(\infty)}]_{i,i+1}.
\end{aligned}$$

which means that

$$\begin{aligned}
[\gamma^{(m)}]_{i,i+r} &= 2^{-T_R(m[D]_{i,i+r})}\left([\varphi^{(m)}]_{i,i+r}\right) \\
&\quad - 2^{-T_R(m[D]_{i,i+r})}\sum_{s=1}^{r}\sum_{k=1}^{s}\left([T^{(m)}\varphi^{(\infty)}]_{i,i+k}[T^{(m)}\varphi^{(\infty)}]^{-1}_{i+k,i+s}[\varphi^{(m)}]_{i+s,i+r}\right) \\
&= 2^{-T_R(m[D]_{i,i+r})}\left([\varphi^{(m)}]_{i,i+r}\right) \\
&\quad - 2^{-T_R(m[D]_{i,i+r})}\sum_{s=1}^{r}\sum_{k=1}^{s}\left([T^{(m)}\varphi^{(\infty)}]_{i,i+k}[T^{(m)}\varphi^{(\infty)}]^{-1}_{i+k,i+s}[\varphi^{(m)}]_{i+s,i+r}\right) \\
&= 2^{-T_R(m[D]_{i,i+r})}\left([\varphi^{(m)}]_{i,i+r} - T^{(m)}_{i,i+r}([\varphi^{(\infty)}]_{i,i+r})\right) \\
&\quad + \ 2^{-T_R(m[D]_{i,i+r})}\sum_{k=1}^{r-1}\left([T^{(m)}\varphi^{(\infty)}]_{i,i+k}\sum_{s=k}^{r}\left([T^{(m)}\varphi^{(\infty)}]^{-1}_{i+k,i+s}[\varphi^{(m)}]_{i+s,i+r}\right)\right).
\end{aligned}$$

Since $|T_R(m[D]_{i,i+k}) + T_R(m[D]_{i+k,i+r}) - T_R(m[D]_{i,i+r})| \leq 1$, we have

$$\begin{aligned}
\|[\gamma^{(m)}]_{i,i+r}\| \leq& 1 + \Bigg\|2^{-T_R(m[D]_{i,i+r})}\left([\varphi^{(m)}]_{i,i+r} - T^{(m)}_{i,i+r}([\varphi^{(\infty)}]_{i,i+r})\right) \\
&\quad + \sum_{k=1}^{r-1}\left(2^{-T_R(m[D]_{i,i+k})}[T^{(m)}\varphi^{(\infty)}]_{i,i+k}[\gamma^{(m)}]_{i+k,i+r}\right)\Bigg\|.
\end{aligned}$$

Combining this result with the triangle inequality given in lemma 2.0.2 and multiplication bound given in 2.0.3,

$$\begin{aligned}
\|[\gamma^{(m)}]_{i,i+r}\| \leq& 1 + 3\left\|2^{-T_R(m[D]_{i,i+r})}\left([\varphi^{(m)}]_{i,i+r} - T^{(m)}_{i,i+r}([\varphi^{(\infty)}]_{i,i+r})\right)\right\| \\
&\quad + 9\sum_{k=1}^{r-1}\left(\left\|2^{-T_R(m[D]_{i,i+k})}\right\| + \left\|[T^{(m)}\varphi^{(\infty)}]_{i,i+k}\right\| + \left\|[\gamma^{(m)}]_{i+k,i+r}\right\|\right).
\end{aligned} \tag{9}$$

Proposition 5.0.7. *Let i and r be natural numbers so that $i < n$ and $r < n - i$. Then*

$$\|[\gamma^{(m)}]_{i,i+r}\| \in o(m).$$

Proof. Proceed by induction. Suppose that

$$\|[\gamma^{(m)}]_{j,j+p}\| \in o(m)$$

for all natural numbers j and r so that $j < n$ and $p < r$. Consider the terms in equation (9). By making similar arguments to those given in the proof of lemma 5.0.6,

$$\left\|2^{-T_R(m[D]_{i,i+r})}\left([\varphi^{(m)}]_{i,i+r} - T^{(m)}_{i,i+r}([\varphi^{(\infty)}]_{i,i+r})\right)\right\| \in o(m).$$

Suppose that k is a natural number less than r. It follows from their definitions that $\left\|2^{-T_R(m[D]_{i,i+k})}\right\|$ and $\left\|[T^{(m)}\varphi^{(\infty)}]_{i,i+k}\right\|$ are also in $o(m)$. The finite sum of these terms in equation (9) must also therefore be in $o(m)$. The inductive assumption gives us that

$$\|[\gamma^{(m)}]_{i+k,i+r}\| \in o(m),$$

hence

$$\|[\gamma^{(m)}]_{k,i+r}\| \in o(m)$$

which completes the proof. ∎

Theorem 5.0.8. *Suppose that μ is a probability measure on G_n with finite first moment and non-zero displacement so that the Poisson boundary of (G_n, μ) is non-trivial (see the conditions given in Proposition 4.2.2). Then, the Poisson boundary of (G, μ) is (B, λ).*

Proof. Recall that

$$\begin{aligned}\left(\pi_m \varphi^{(\infty)}\right)^{-1}(\boldsymbol{y}^{(m)}, \varphi^{(m)}) &= ((\boldsymbol{t}^{(m)})^{-1}\boldsymbol{x}^{(m)}, (\boldsymbol{t}^{(m)})^{-1}(T^{(m)}\varphi^{(\infty)})^{-1}\boldsymbol{t}^{(m)}(\boldsymbol{t}^{(m)})^{-1}\varphi^{(m)}\boldsymbol{t}^{(m)}) \\ &= ((\boldsymbol{t}^{(m)})^{-1}\boldsymbol{x}^{(m)}, (\boldsymbol{t}^{(m)})^{-1}(T^{(m)}\varphi^{(\infty)})^{-1}\varphi^{(m)}\boldsymbol{t}^{(m)}) \\ &= ((\boldsymbol{t}^{(m)})^{-1}\boldsymbol{x}^{(m)}, \gamma^{(m)}).\end{aligned}$$

The previous proposition, lemma 5.0.5, and the fact that

$$|x_i^{(m)} - T_R(m\overline{\mu_i})| \in o(m) \ \mathbb{P}^\mu\text{–a.s.}$$

imply that

$$|\left(\pi_m \varphi^{(\infty)}\right)^{-1}(\boldsymbol{y}^{(m)}, \varphi^{(m)})|_{K_n} \leq C_n \left(\sum_{i=1}^{n} |x_i^{(m)} - T_R m\overline{\mu_{x_i}}| + \sum_{i=1}^{n-1}\sum_{r=1}^{n-i} \|[\gamma^{(m)}]_{i,i+r}\|\right) \in o(m).$$

Thus, (B, λ) is the Poisson boundary by Kaimanovich's ray criterion. □

We remark that, for p prime, if we had instead defined G_n to be the group of upper triangular $n \times n$ matrices whose entries are integer powers of p on the main diagonal and $\frac{a}{p^b}$ for some integers a and b in each off upper diagonal, then all the arguments used in the $p = 2$ case, with the necessary modifications, still apply in the description of the Poisson boundary. In particular, dependant on the displacement(s), each 2-adic component is instead a p-adic component.

5.1. **The Poisson boundary of the difference subgroup.** Suppose that K_n is any subgroup of N_n and let F_n be the subgroup $\mathbb{Z}^n \ltimes K_n$ of G_n. Let $F_n/\ker\Delta$ be the difference subgroup of F_n as described in 1.1. Suppose that μ is a probability measure on F_n which has finite first moment.

Let D be the $n \times n$ mean displacement matrix determined by the values

$$[D]_{k,k+1} \coloneqq \sum_{z\in\mathbb{Z}} z\mu_k(z)$$

where k is a natural number less than n and μ_k is the projection of μ on the kth integer factor of F_n on to $\mathbb{Z}$.

Then the Poisson boundary of (F_n, μ) for measures μ with non-zero displacements(s) may be described using, *mutatis mutandis*, the arguments given to describe the boundary of G_n for those measures which satisfy the same conditions. The necessary and sufficient conditions for boundary may also be determined similarly.

5.2. **The Poisson boundary of the subnormal quotients of G_n.** Describing the Poisson boundary of the quotients $G_n^{(j)}/G_n^{(k)}$ for non-negative integers j and k is also of interest. Any quotient $G_n^{(j)}/G_n^{(k)}$ such that $0 < j < k$ is nilpotent and so has trivial boundary for every measure. The groups $G_n/G_n^{(0)}$ and $G_n/G_n^{(1)}$ are respectively trivial and abelian, so the Poisson boundaries of those groups are also always trivial.

Suppose now that $k > 1$ is fixed and that μ is a probability measure on $G_n/G_n^{(k)}$ with finite first moment. Then $G_n/G_n^{(k)}$ is solvable, but not nilpotent, so it may have a non-trivial boundary. Recall that the quotients $G_n/G_n^{(k)}$ for $k > 1$ are the cosets of G_n for which any two elements (x, f) and (y, g) are identified if $x = y$ and any differences in f and g occur above the first k upper super-diagonals. Defining the displacement matrix D in the same way as for G_n, the entries in the first k upper super-diagonals of the coset representatives will converge in either $\mathbb{R}$ or $\mathbb{Q}_2$ depending on the sign of the displacements.

Let B be the set of all $n \times n$ matrices M whose entries $[M]_{ij}$ are respectively in $\mathbb{R}$ or $\mathbb{Q}_2$ if $[D]_{ij} < 0$ or $[D]_{ij} > 0$ where two elements M and N are identified if any differences above the first k upper super-diagonals. Regard B as a $G_n/G_n^{(k)}$-space under the action of left multiplication. It follows from arguments made for G_n that B is a boundary of $(G_n/G_n^{(k)}, \mu)$ with the hitting measure λ. Kaimanovich's ray criterion may be used to show maximality by choosing coset representatives in the same way that elements were chosen for G_n.

The Poisson boundary for measures of finite first moment and non-zero displacement on the subnormal series of $G_n/\ker\Delta$ may also be described similarly.

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